Fuzzy Description Logic Programs under the Answer Set Semantics for the Semantic Web

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Motivation

Ingredients:

- Expressive description logics behind OWL Lite and OWL DL ($\mathcal{SHIF}(\mathbf{D})$ resp. $\mathcal{SHOIN}(\mathbf{D})$).
- Rule-based formalism (normal logic programs under the answer set semantics).
- Fuzzy truth functions (for conjunction and negation).

Motivation:

- Fuzzy query language for multimedia databases, containing images and videos (such as Google's YouTube), along the lines of "R. Fagin. Fuzzy queries in multimedia database systems. In *Proceedings PODS-1998*".
- Expressing vague terms in natural language interfaces to the Web / Semantic Web.

Outline

$PC \sqcup Camera \sqsubseteq Electronics$; $PC \sqcap Camera \sqsubseteq \bot$;
$Book \sqcup Electronics \sqsubseteq Product$; $Book \sqcap Electronics \sqsubseteq \bot$;
$Textbook \sqsubseteq Book$;

$Product \sqsubseteq \geq 1\, related$;
$\geq 1\, related \sqcup \geq 1\, related^{-} \sqsubseteq Product$;

$Textbook(tb\_ai)$; $Textbook(tb\_lp)$;
$PC(pc\_ibm)$; $PC(pc\_hp)$;

$related(tb\_ai, tb\_lp)$; $related(pc\_ibm, pc\_hp)$;
$provides(ibm, pc\_ibm)$; $provides(hp, pc\_hp)$.

Finite set of truth values $TV = \{\frac{0}{n}, \frac{1}{n}, \ldots, \frac{n}{n}\}$ with $n \geq 1$.

A fuzzy atomic concept assertion has the form $C(a) \geq v$, where $C \in \mathbf{A}$, $a \in \mathbf{I}$, and $v \in TV$. A fuzzy abstract (resp., datatype) role assertion has the form $R(a, b) \geq v$ (resp., $U(a, s) \geq v$), where $R \in \mathbf{R}_A$ (resp., $U \in \mathbf{R}_D$), $a, b \in \mathbf{I}$ (resp., $a \in \mathbf{I}$, and s is a data value), $v \in TV$.

A fuzzy description logic knowledge base $KB = (L, F)$ consists of an ordinary description logic knowledge base L and a finite set of fuzzy atomic concept assertions and fuzzy role assertions F.

Example: A simple fuzzy description logic knowledge base $KB = (L, F)$ is given by L above and

$$F = \{\mathit{Inexpensive}(\mathit{pc\_ibm}) \geq 0.6,\ \mathit{Inexpensive}(\mathit{pc\_hp}) \geq 0.9\}\,.$$

Here, F encodes the different degrees of membership of PCs by IBM and HP to the fuzzy concept *Inexpensive*.

The ordinary equivalent to a set of fuzzy concept and role assertions F, denoted $F^\star$, is obtained from F by replacing each $C(a) \geq v$ (resp., $R(a,b) \geq v$, $U(a,s) \geq v$) by $C^v(a)$ (resp., $R^v(a,b)$, $U^v(a,s)$).

The v-layer of L, denoted L^v, is obtained from L by replacing every $C \in \mathbf{A}$ (resp., $R \in \mathbf{R}_A$, $U \in \mathbf{R}_D$) by C^v (resp., R^v, U^v).

The ordinary equivalent to a fuzzy description logic knowledge base $KB = (L, F)$, denoted $KB^\star$, is defined as

$$\bigcup_{v \in TV,\, v>0} L^v \cup F^\star \cup \{A^v \sqsubseteq A^{v'} \mid A \in \mathbf{A}, v \in TV, v \geq 2/n, v' = v-1/n\} \cup$$
$$\{R^v \sqsubseteq R^{v'} \mid R \in \mathbf{R}_A, v \in TV, v \geq 2/n, v' = v-1/n\} \cup$$
$$\{U^v \sqsubseteq U^{v'} \mid U \in \mathbf{R}_D, v \in TV, v \geq 2/n, v' = v-1/n\}\,.$$

KB is satisfiable iff $KB^\star$ is satisfiable.

F among $C(a) \geq v$, $R(a,b) \geq v$, and $U(a,s) \geq v$ is a logical consequence of KB, denoted $KB \models F$, iff $C^v(a)$, $R^v(a,b)$, and $U^v(a,s)$, respectively, are logical consequences of $KB^\star$.

Finite set of truth values $TV = \{\frac{0}{n}, \frac{1}{n}, \ldots, \frac{n}{n}\}$ with $n \geq 1$.

Negation strategies $\ominus$: $TV \rightarrow TV$ such that
$\ominus$ is antitonic and satisfies $\ominus\, 0 = 1$ and $\ominus\, 1 = 0$.

Example: $\ominus\, v = 1 - v$.

Conjunction strategies $\otimes$: $TV \times TV \rightarrow TV$ such that
$\otimes$ is commutative, associative, monotonic, and
satisfies $v \otimes 1 = v$ and $v \otimes 0 = 0$.

Example: $v_1 \otimes v_2 = \min(v_1, v_2)$ and $v_1 \otimes v_2 = v_1 \cdot v_2$.

A normal fuzzy rule r is of form (with atoms $a, b_1, \ldots, b_m$):

$$\begin{aligned} a \leftarrow_{\otimes_0} b_1 \wedge_{\otimes_1} b_2 \wedge_{\otimes_2} \cdots \wedge_{\otimes_{k-1}} b_k \wedge_{\otimes_k} \\ not_{\ominus_{k+1}} b_{k+1} \wedge_{\otimes_{k+1}} \cdots \wedge_{\otimes_{m-1}} not_{\ominus_m} b_m \geq v, \end{aligned} \qquad (1)$$

A normal fuzzy program P is a finite set of normal fuzzy rules.

A dl-query $Q(\mathbf{t})$ is of one of the following forms:

- a concept inclusion axiom F or its negation $\neg F$;
- $C(t)$ or $\neg C(t)$, with a concept C and a term t;
- $R(t_1, t_2)$ or $\neg R(t_1, t_2)$, with a role R and terms t_1, t_2.

A fuzzy dl-rule r is of form (1), where any $b \in B(r)$ may be a dl-atom, which is of form $DL[S_1 op_1 p_1, \ldots, S_m op_m\, p_m; Q](\mathbf{t})$.

A fuzzy dl-program $KB = (L, P)$ consists of a description logic knowledge base L and a finite set of fuzzy dl-rules P.

(1) $pc(pc\_1) \geq 1$; $pc(pc\_2) \geq 1$; $pc(pc\_3) \geq 1$;

(2) $brand\_new(pc\_1) \geq 1$; $brand\_new(pc\_2) \geq 1$;

(3) $offer(X) \leftarrow_{\otimes} DL[PC \uplus pc; Electronics](X) \wedge_{\otimes}$
$not_{\ominus}\, brand\_new(X) \geq 1$;

(4) $buy(C, X) \leftarrow_{\otimes} needs(C, X) \wedge_{\otimes} offer(X) \geq 0.7$;

(5) $buy(C, X) \leftarrow_{\otimes} needs(C, X) \wedge_{\otimes} DL[Inexpensive](X) \geq 0.3$.

(4) A customer who needs a product on offer buys this product with degree of truth of at least 0.7.

(5) A customer who needs an inexpensive product buys this product with degree of truth of at least 0.3.

$\ominus$ and $\otimes$ are given by $\ominus\, v = 1 - v$ and $v_1 \otimes v_2 = \min(v_1, v_2)$.

An interpretation I (relative to P) is a mapping $I\colon HB_P \rightarrow TV$.

The truth value of $a = DL[S_1 \uplus\ p_1, \ldots, S_m \uplus p_m; Q](\mathbf{c})$ under L, denoted $I_L(a)$, is defined as the maximal truth value $v \in TV$ such that $L \cup \bigcup_{i=1}^{m} A_i(I) \models Q(\mathbf{c}) \geq v$, where

$$A_i(I) = \{S_i(\mathbf{e}) \geq I(p_i(\mathbf{e})) \mid I(p_i(\mathbf{e})) > 0,\ p_i(\mathbf{e}) \in HB_P\}\,.$$

I is a model of a ground fuzzy dl-rule r of the form (1) under L, denoted $I \models_L r$, iff

$$\begin{aligned} I_L(a) \geq v \otimes_0 I_L(b_1) \otimes_1 I_L(b_2) \otimes_2 \cdots \otimes_{k-1} I_L(b_k) \otimes_k \\ \ominus_{k+1} I_L(b_{k+1}) \otimes_{k+1} \cdots \otimes_{m-1} \ominus_m I_L(b_m), \end{aligned}$$

I is a model of a fuzzy dl-program $KB = (L, P)$, denoted $I \models KB$, iff $I \models_L r$ for all $r \in ground(P)$.

A fuzzy dl-program $KB = (L, P)$ is positive iff P is "*not*"-free.

Theorem: Positive fuzzy dl-programs KB are satisfiable and have a unique least model, denoted M_{KB}, as a natural semantics.

Example: Consider the fuzzy dl-program KB consisting of the above fuzzy description logic knowledge base and the fuzzy dl-rules

(0) $needs(john, pc\_ibm) \geq 1$;

(1) $pc(pc\_1) \geq 1$; $pc(pc\_2) \geq 1$; $pc(pc\_3) \geq 1$;

(2) $brand\_new(pc\_1) \geq 1$; $brand\_new(pc\_2) \geq 1$;

(4) $buy(C, X) \leftarrow_{\otimes} needs(C, X) \wedge_{\otimes} offer(X) \geq 0.7$;

(5) $buy(C, X) \leftarrow_{\otimes} needs(C, X) \wedge_{\otimes} DL[Inexpensive](X) \geq 0.3$.

Then, KB is positive, and $M_{KB}(buy(john, pc\_ibm)) = 0.3$.

Stratified fuzzy dl-programs are composed of hierarchic layers of positive fuzzy dl-programs linked via default negation:

A stratification of $KB = (L, P)$ with respect to DL_P is a mapping $\lambda : HB_P \cup DL_P \rightarrow \{0, 1, \ldots, k\}$ such that

- $\lambda(H(r)) \geq \lambda(a)$ (resp., $\lambda(H(r)) > \lambda(a)$) for each $r \in ground(P)$ and $a \in B^+(r)$ (resp., $a \in B^-(r)$), and
- $\lambda(a) \geq \lambda(a')$ for each input atom a' of each $a \in DL_P$,

where $k \geq 0$ is the length of λ. A fuzzy dl-program $KB = (L, P)$ is stratified iff it has a stratification λ of some length $k \geq 0$.

Theorem: Every stratified fuzzy dl-program KB is satisfiable and has a canonical minimal model via a finite number of iterative least models (which does not depend on the stratification of KB).

Example: Consider the fuzzy dl-program *KB* consisting of the above fuzzy description logic knowledge base and the fuzzy dl-rules

(0) $needs(john, pc\_ibm) \geq 1$;

(1) $pc(pc\_1) \geq 1$; $pc(pc\_2) \geq 1$; $pc(pc\_3) \geq 1$;

(2) $brand\_new(pc\_1) \geq 1$; $brand\_new(pc\_2) \geq 1$;

(3) $offer(X) \leftarrow_{\otimes} DL[PC \uplus pc; Electronics](X) \wedge_{\otimes}$
$not_{\ominus}\, brand\_new(X) \geq 1$;

(4) $buy(C, X) \leftarrow_{\otimes} needs(C, X) \wedge_{\otimes} offer(X) \geq 0.7$;

(5) $buy(C, X) \leftarrow_{\otimes} needs(C, X) \wedge_{\otimes} DL[Inexpensive](X) \geq 0.3$.

Then, *KB* is stratified, and it holds in particular
$M_{KB}(offer(pc\_ibm)) = 1$ and $M_{KB}(buy(john, pc\_ibm)) = 0.7$.

Let $KB = (L, P)$ be a fuzzy dl-program. The fuzzy dl-transform of P relative to L and an interpretation $I \subseteq \boldsymbol{HB}_P$, denoted P_L^I, is the set of all fuzzy dl-rules obtained from $ground(P)$ by replacing all default-negated atoms $not_{\ominus_j} a$ by the truth value $\ominus_j I_L(a)$.

An answer set of KB is an interpretation $I \subseteq \boldsymbol{HB}_P$ such that I is the least model of (L, P_L^I).

Theorem: Let KB be a fuzzy dl-program, and let M be an answer set of KB. Then, M is a minimal model of KB.

Theorem: Let KB be a positive (resp., stratified) fuzzy dl-program. Then, M_{KB} is its only answer set.

For a fuzzy dl-program $KB = (L, P)$, define the operator T_{KB} as follows. For every $I \subseteq \boldsymbol{HB}_P$ and $a \in HB_P$, let $T_{KB}(I)(a)$ be defined as the maximum of v subject to $r \in ground(P)$, $H(r) = a$, and v being the truth value of r's body under I and L. Note that if there is no such rule r, then $T_{KB}(I)(a) = 0$.

Lemma: Let $KB = (L, P)$ be a positive fuzzy dl-program. Then, the operator T_{KB} is monotonic.

Theorem: Let $KB = (L, P)$ be a positive fuzzy dl-program. Then, $lfp(T_{KB}) = M_{KB}$. Furthermore,

$$lfp(T_{KB}) = \bigcup_{i=0}^{n} T_{KB}^{i}(\emptyset) = T_{KB}^{n}(\emptyset), \text{ for some } n \geq 0.$$

M_{KB} of a stratified fuzzy dl-program KB can be characterized by a sequence of fixpoint iterations along a stratification:

Let $\widehat{T}^i_{KB}(I) = T^i_{KB}(I) \cup I$, for all $i \geq 0$.

Theorem: Let $KB = (L, P)$ be a fuzzy dl-program with stratification λ of length $k \geq 0$. Let $M_i \subseteq \boldsymbol{HB}_P$, $i \in \{-1, 0, \ldots, k\}$, be defined by $M_{-1} = \emptyset$, and $M_i = \widehat{T}^{n_i}_{KB_i}(M_{i-1})$ for every $i \geq 0$, where n_i such that $\widehat{T}^{n_i}_{KB_i}(M_{i-1}) = \widehat{T}^{n_i+1}_{KB_i}(M_{i-1})$. Then, $M_k = M_{KB}$.

Summary:

- Simple fuzzy extensions of $\mathcal{SHIF}(\mathbf{D})$ and $\mathcal{SHOIN}(\mathbf{D})$.
- Unique least model and iterative least model semantics of positive resp. stratified fuzzy dl-programs.
- Answer set semantics of general fuzzy dl-programs. Coincides with the canonical semantics in the positive and stratified case.
- Fixpoint and iterative fixpoint characterization of the canonical semantics of positive resp. stratified fuzzy dl-programs.

Outlook:

- Computational complexity, efficient algorithms (especially for general fuzzy dl-programs), and implementation.
- Integration of more expressive fuzzy description logics.